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Reports and Comments

The Use of the Production Function and Linear Programming in Valuation of Intermediate Products

Introduction

THE valuation of intermediate products has long been a difficult problem in applied economics. Theoretically, the problem is solved by the equation of the marginal value productivity of a factor with its price. For intermediate products it is difficult to obtain accurately marginal value product functions; sometimes the price is also unknown. However, empirical estimates of these magnitudes are needed if the problem is to have a solution. In the problem to be investigated this information is important to both the individual farm manager and the public lands administrator.

The essentials of the problem were set out by Johnson and Hardin[1] in a paper dealing with forage valuation. Their frame of reference was to take acquisition value as the upper limit in valuation and salvage value as the lower limit; between these extremes the marginal productivity value is the relevant magnitude for decision making.

The problem with which this paper is concerned is to estimate the most profitable level of nitrogen fertilizer use in production of wild hay and pasture from mountain meadows.[2] To obtain such an estimate it was necessary to value hay; in the process other inputs were valued. This by-product has interesting implications regarding the administration of public range lands. This paper is organized as follows: (1) a review of alternative methods of resource valuation, (2) a demonstration of the use of a production function in setting up a programming model and (3) valuation of hay, public grazing land and pasture by the use of linear programming.

Methods of Resource Valuation

A commonly used method of valuation is residual imputation. This procedure involves the imputation of the total physical or value product. When this procedure is used the accounting costs of all inputs except the one being valued is subtracted from gross income. The residual is the value attributed to the input in question. The assumptions of this procedure are that the market price of each resource equals its marginal value product and no residual can remain when each factor is imputed its exact reward expressed in terms of market prices. This procedure has two disadvantages—first, the marginal value product does not always equal the market price and second, there is the problem of imputing a return to management and unpaid family labor.[3] Because of the arbitrary assumptions associated with this method it was considered unsatisfactory for the problem at hand.

A second method is the use of multiple regression analysis. The Cobb-Douglas function is the one most commonly used in this type of analysis. This procedure yields the elasticities of the various factors of production and from these the marginal value productivities may be calculated. The method assumes that the products can be aggregated into a single dependent variable expressed in money terms and the inputs can be aggregated into different independent variables. The disadvantages and problems associated with these assumptions have been pointed out by Plaxico,[4] and they seriously

[1] Lowell S. Hardin and Glenn L. Johnson, "Economics of Forage Evaluation," *Journal of Farm Economics*, December 1955, pp. 1457-1459.

[2] The Squaw Butte-Harney Experiment Station carried out trials in 1954 and 1955 to test the hay yield response to nitrogen fertilizer on meadow land in the Harney basin, Oregon, which floods for 8-12 weeks in spring.

[3] E. O. Heady, *Economics of Agricultural Production and Resource Use* (New York: Prentice-Hall, Incorporated, 1952), pp. 407-408.

[4] James S. Plaxico, "Problems of Factor-Product Aggregation in Cobb-Douglas Value Production by Analysis," *Journal of Farm Economics*, November 1955, pp. 664-675.

limit the usefulness of Cobb-Douglas estimates as guides for intra-farm decisions.

In recent years linear programming has come into considerable popularity as a method of maximizing returns to certain limiting factors. Although Dorfman[5] and others have pointed out that resource valuation is implicit in the method, its possibilities in this respect have not been adequately explored in empirical work. The method permits a simultaneous valuation of the resources used in production. In an actual production process this is, of course, what happens. Euler's theorem states this more precisely—if each factor is imputed its marginal product, the total product will be exactly exhausted if the condition of constant returns to scale is fulfilled. Euler's theorem forms the basis of one of the assumptions of linear programming, that of the linear relationship or constant returns to scale. Other assumptions are that the processes available are finite and that the processes are independent, additive and divisible. Since linear programming and multiple regression analysis both permit the rewards to each factor to be determined simultaneously, these models are different from the accounting approach involving the residual imputation procedure in which all but one of the factors must already be valued in order to obtain a solution. As stated above, the Cobb-Douglas function is not generally satisfactory for intra-farm analysis and resource valuation. In cases where the problem and data available are such as to permit the use of the linear programming technique, it would seem that simultaneous solution of rewards to factors of production is more logical than the residual imputation procedure.

Time is also a problem in resource valuation. This problem may be dealt with by discounting expected future revenue and compounding expected future costs. Normally, however, there is not satisfactory empirical data which might provide a basis for estimating both the rate of interest and the time period which should be used. The usual procedure is to apply the current rate of interest and use a time span of 20 to 25 years. Another aspect of this problem is that of adjusting values for risk and uncertainty. Allowance for risk may be taken as the cost of insurance, but there is again no satisfactory standard from which the discount rate for uncertainty may be established. This paper does not introduce risk and uncertainty into the programming model in a formal fashion. The results, however, are interpreted in the knowledge that risk and uncertainty actually exist in the problem situation.

The Hay Production Function

As already stated, the problem towards which the study was directed was the estimation of the economic level of nitrogen fertilizer use in wild hay production from native mountain meadows.

To obtain the hay yield coefficient at any given level of nitrogen it was necessary to construct a production function from the experimental data.[6] In order to select the appropriate regression equation, five expressions representing three types of function were applied to the experimental results. A comparison of these expressions is given in Table I. In terms of biological logic it is difficult to attribute any significance to the powers in the power functions; so, in the absence of any strong statistical reasons for the

[5] Robert Dorfman, *Application of Linear Programming to the Theory of the Firm* (Berkeley: University of California Press, 1951).

[6] Fertilizer trials carried out by the Squaw Butte-Harney Experiment Station. All other coefficients were obtained from a survey of ranchers in the area and market reports.

Table 1.—Comparison of Goodness of Fit Obtained by Use of Different Prediction Equations

Type of Function		Expression	$\Sigma(y-\hat{y})$	$\Sigma(y-\hat{y})^2$	Estimating equation
Exponential		$\hat{y} = M(1 - R^x)$	-.021667	.012712	$Y = 3.851755 (1 - .8^{2.892581 \; .044435x})$
Power	(1)	$\hat{y} = ab^x x$	.015370	.013241	$Y = 0.018244 (.99635)^x x$
	(2)	$\hat{y} = ab_1^x x^{b2}$	-.00370	.000967	$Y = 0.71025 (.99981)^x x^{0.618845}$
Polynomial	(1)	$\hat{y} = a + b_1 x + b_2 \sqrt{x}$	.000001	.000953	$Y = 1.830943 + .002205x + .097691\sqrt{x}$
	(2)	$\hat{y} = a + b_1 x - b_2 x^2$	-.000090	.015054	$Y = 1.871226 + .014946x - .000031016x^2$

selection of one expression, biological theories would tend to support the use of either the exponential or the polynomial form. It can be seen from columns 3 and 4 in Table I that the power functions are not superior to the other functions in goodness of fit as measured by the minimum sum of squares. Therefore, the two power functions were eliminated. The minimum sum of squares was used as the basis for selection among the remaining equations. From Table I it will be seen that the polynomial with the form, $y = a + b_1 + b_2$ gives the lowest sum of squares, .000953. But both the b coefficients in this expression are positive, thus giving a continuously increasing function over the entire growth range, which cannot be justified on the grounds of biological theory. Of the two remaining expressions, the exponential, $y = M(1-R^x)$, has the lowest sum of squares. On this basis, it was selected for use in this analysis.

The total product function was used in the linear programming work described in the next section. Points on the production surface ranging from 0 to 160 pounds of nitrogen were selected. The points were 0, 40, 60, 80, 100, 120 and 160. Each point was selected as a "process" in the matrix. When a process, say, 60 pounds of nitrogen and 2.5 tons of hay, was "selected," further refinement was obtained by including 50 pounds of nitrogen as "processes." If 50 was selected over 60, than we know 50 is to be preferred to either 40 or 60. In this particular problem great refinement in this respect was not warranted. In the next section the programming method is described in greater detail.

The Linear Programming Model

The first step in setting up the initial matrix was to isolate the relevant factors affecting the decision on fertilizer use. A survey was made of the ranches in the area and resource situations in terms of land, labor and capital were determined. The coefficients used in constructing the matrices were obtained from the survey.

A total of three matrices were constructed and solved to obtain optimum fertilizer use under different assumed conditions. The linear programming technique permitted the level of fertilization, the level of beef production and the acreage devoted to different uses to be determined simultaneously. Each "model" is described briefly below.

MATRIX I: This is a two-man unit with six limiting resources. No fertilizer activities were included for reasons that will be apparent later. The range permit was 3,025 animal unit months, the base property consisted of 750 acres of flood meadow, of which 260 acres (Meadow II) gave unsatisfactory response to fertilizer because of deep swales and excess alkalinity of the soil. This area yielded 1 ton of wild hay per acre. The remaining 490 acres (Meadow I) gave a yield of 1.2 tons per acre without fertilizer. This matrix serves as a "check" with which the other matrices can be compared. This might be classified as a "typical" situation without the use of fertilizer.

MATRIX II: This matrix was identical to Matrix I except that fertilizer activities were introduced. Different levels of fertilization were included as processes. This permitted additional pasture and hay to be produced which could be substituted for range land. It also permitted the economic rate of fertilization to be selected.

MATRIX III: Range was permitted to become unlimited in this matrix, the purpose being to indicate the potential of range improvement. In this case there were four limitational resources, Meadows I and II, stacked hay, bunched hay, and four levels of nitrogen on each of the two forage activities. Any additional capital required for the system was assumed to be available at 7 percent interest. In interpreting the data below, it should be borne in mind that bunched hay and stacked hay are fed in combination. All coefficients used in the matrix with the exception of the fertilizer response data were obtained from a survey of the ranchers of the area. In all cases 1956 prices and costs were used.

Optimum Fertilization Rates

The programming model permitted determination of the acreage to be fertilized and the optimum rate of fertilization. The results are presented in Table II. Matrix II shows

TABLE II—FERTILIZER RATES, LAND USE AND BEEF PRODUCTION WITH VARYING RESOURCE SITUATIONS

	Matrix II	Matrix III
Stacked hay......	282 acres at 50# N	313 acres at 100# N
Bunched hay.....	118 acres at 40# N	177 acres at 90# N
Meadow pasture..	90 acres at 50# N	....
Increase in beef production over Matrix I.......	26%	66%
Increase in net return over Matrix I.........	$1,443.64	$3,782.13

that a typical ranch situation can profitably apply from 40 to 50 pounds of nitrogen to meadow and pasture. This will permit the additional forage produced to be partially substituted for the public range land.

Of considerable interest is the extent of the limitation imposed by the range resource. The solution of Matrix III indicates that beef production could be expanded by 66 percent if range land were not a limiting factor. This emphasizes the need for an economical method of range improvement and underlines the importance of the work being done by the physical scientists in this area.

Resource Valuation

A by-product obtained from the solution of these models was the valuation of the marginal products. This is the familiar case of the "dual problem" as it has been labeled by Dorfman. These marginal productivity values are presented in Table III.

TABLE III—Marginal Productivity Values of Resources

Matrix Number	Ton of Stacked Hay	Ton of Bunched Hay	Per AUM of Meadow Pasture	Per AUM of Range
I.......	10.81	9.64	......	5.17
II......	14.06	12.81	3.06	4.51
III.......	24.13	22.77	......	......

The reader may be surprised at the productivity values when it is realized the marginal productivity of hay increases with heavier rates of nitrogen application. Upon consideration, however, one familiar with the programming technique will not be dismayed. In Matrix I the marginal value productivity of hay is lowest because of the "tight" range limitation. However, in Matrix II where fertilization of hay and pasture is allowed, it becomes profitable to substitute increased hay and pasture for range which "loosens" the range restriction and results in higher hay MVP's. Finally, in Matrix III where range is unlimited it pays to increase production of hay to a point indicating a very high marginal productivity value to utilize the feeding value of the relatively cheap range. Hay becomes very valuable in this last case because of the complementary relationship between hay and range; some hay is required for beef production for wintering even though range is utilized to the utmost extent.

Conversely, the MVP's of range are lowered as the quantity of relatively cheap range is allowed to increase. In Matrix I range is very much a limiting factor and the MVP is high. In Matrix II it is less limiting as hay and pasture is substituted for it. Finally, when range becomes unlimited, its MVP goes to zero; the rancher would be unwilling to pay anything for an additional unit.

It is possible the marginal productivity values are slightly overestimated. Some overhead costs that do not change with fertilization were omitted from the analysis. Such costs, however, are quite minor. It is interesting to note that a minor amount of hay was sold in the area in 1956 which ranged in price from $12 to $15 per ton.

The discrepancy between these estimates, $5.17 and $4.51, and the rent actually charged by the Bureau of Land Management needs explanation. This charge varies from $0.15 to $0.44 per AUM. In the first place, the estimate of $4 to $5 per AUM should be discounted for risk and uncertainty. Because of the high degree of risk and uncertainty associated with the desert cattle operation, the discount rate could well be as high as 50 percent. If this were the case, the estimate would be reduced to $2 to $2.50. The historical aspect is also a factor in explaining this discrepancy. Originally the range land was free, and when it came under administration of the Forest Service and Bureau of Land Management, neither agency felt justified in charging the full market price for grazing rights. This feeling still prevails. Originally, of course, when an abundance of range existed, its value was much lower than at present when it is very much a limiting resource. It is also likely that the value of the range land has increased over time and it is doubtful that prices charged by the government agencies have kept pace.

It should be pointed out, however, that the discrepancy does not necessarily mean a subsidy to present operators. The excess value of the range permit has been capitalized back into the base acreage in many cases. The purchaser of a range unit at the present time would be paying for this surplus in advance. Many of the present operators have already paid for this excess value over cost. Whenever a discrepancy of this kind occurs, however, definite administrative problems are created. While this study was not mainly concerned with policy problems, certain policy implications are apparent.

Conclusions

Where suitable input-output data are available, linear programming provides a method of valuation for productive inputs. It overcomes many of the limitations of other methods. One problem in the use of this technique is the introduction of sufficient subjectivity, which can be readily incorporated in budgeting. Linear programming may often be too complicated or cumbersome to replace budgeting in an economic analysis. However, it has a superior feature in that it permits simultaneous determination of the optimum level of output, the combination of processes to be used, and the valuation of the limiting resources.

In the research reported in this paper the optimum level of beef production and the economic rate of fertilization were determined. In addition, values of hay, pasture and range land were obtained. The hay values are of primary interest to the individual rancher who must choose between fertilization and other methods of obtaining hay. The value of the range is of interest to those involved in research on and management of publicly-owned resources.

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